

Hausdorff property of the Néron models of Green, Griffiths and Kerr

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Abstract. We prove the Hausdorff property of the Néron model of the family of intermediate Jacobians which is recently defined by Green, Griffiths and Kerr assuming that the divisor at infinity is smooth. Using their result, this implies in this case the analyticity of the closure of the zero locus of an admissible normal function. The last assertion is also obtained by Brosnan and Pearlstein generalizing their method in the curve case.

Introduction

Let \mathbf{H} be a polarizable variation of \mathbf{Z} -Hodge structure of weight $w < 0$ on a complex manifold S . Let $J_S(\mathbf{H})$ denote the family of intermediate Jacobians over S . A *normal function* is a holomorphic section of $J_S(\mathbf{H})$ satisfying the Griffiths transversality, and corresponds by a well-known theorem of Carlson [3] to an extension class of the trivial variation \mathbf{Z}_S by \mathbf{H} in the abelian category of mixed \mathbf{Z} -Hodge structures [6]. A normal function is called *admissible* [10] if the corresponding extension is an admissible variation of mixed Hodge structure in the sense of [9], [13]. This condition corresponds to certain conditions on the local monodromy and the logarithmic growth at infinity of the normal function, see [8]. By M. Green and P. Griffiths (see [1]) we have the following

Conjecture 1. The zero locus of an admissible normal function is algebraic if S is algebraic (where $w = -1$).

Let \bar{S} be a smooth partial compactification of S such that $D := \bar{S} \setminus S$ is a divisor with normal crossings. Then Conjecture 1 is easily reduced to

Conjecture 2. The closure of the zero locus of an admissible normal function in \bar{S} is analytic (where $w = -1$).

In the curve case these conjectures were solved by P. Brosnan and G. Pearlstein [1] where the assertion was equivalent to the discreteness of the closure. In this paper we prove

Theorem 1. *Assume D is smooth. If $w = -1$, then Conjecture 2 is true, and moreover, the closure of the zero locus is contained in S if the local cohomological invariant of the admissible normal function is nonzero and the local monodromy of \mathbf{H} is unipotent. The*

same assertions hold for $w < -1$ if we assume further that the local cohomological invariant of the admissible normal function is torsion.

We are informed that the first assertion is also obtained by P. Brosnan and G. Pearlstein generalizing their method in [1], see [2]. From Theorem 1 we can deduce

Corollary 1. *Assume S admits a smooth compactification \bar{S} such that $D = \bar{S} \setminus S$ is smooth. Then Conjecture 1 holds where $w = -1$. For $w < -1$ it holds if furthermore the local cohomological invariant of the admissible normal function is torsion.*

The proof of Theorem 1 is reduced to the case where the local monodromy is unipotent. Let $J_{\bar{S}}^Z(\mathbf{H})$ denote the Zucker extension of $J_S(\mathbf{H})$, see [14]. Let $J_{\bar{S}}^{\text{GK}}(\mathbf{H})^0$ denote the subspace of $J_{\bar{S}}^Z(\mathbf{H})|_D$ defined by the quotient of the monodromy invariant subspace. This is the identify component of the Néron model $J_{\bar{S}}^{\text{GK}}(\mathbf{H})$ which is recently defined by M. Green, P. Griffiths and M. Kerr, see [8]. Using their result, Theorem 1 is then reduced to the following

Theorem 2. *The Zucker extension $J_{\bar{S}}^Z(\mathbf{H})$ has a structure of a complex Lie group over \bar{S} , and it is a Hausdorff topological space on a neighborhood of $J_{\bar{S}}^{\text{GK}}(\mathbf{H})^0$.*

This is essentially equivalent to Corollary (2.7) which is deduced from Theorem (2.1). The latter is proved by generalizing some arguments in [10] used to define the structure of a complex Lie group on the Zucker extension $J_{\bar{S}}^Z(\mathbf{H})$ in the curve case. The proof consists of elementary calculations for norm estimates. Note that the Zucker extension outside the monodromy invariant subspace is not necessarily Hausdorff, see e.g. [10], Remark 3.5 (iv). As a corollary we get

Theorem 3. *The Néron model of Green, Griffiths and Kerr $J_{\bar{S}}^{\text{GK}}(\mathbf{H})$ is Hausdorff where $w = -1$.*

In Section 1 we review some basic facts from the theory of Zucker extensions, admissible normal functions and limit mixed Hodge structures, and prepare some notation. In Section 2 we first show Theorem (2.1) on the norm estimate, and then prove Theorems 1–3.

1. Preliminaries

In this section we review some basic facts from the theories of Zucker extensions, admissible normal functions, and limit mixed Hodge structures, and prepare some notation.

1.1. Family of intermediate Jacobians. Let S be a complex manifold of dimension n , and let $\mathbf{H} = ((\mathcal{L}, F), L_{\mathbf{Z}})$ be a polarizable variation of \mathbf{Z} -Hodge structure of weight $w < 0$ on S . Here (\mathcal{L}, F) is the underlying filtered locally free sheaf and $L_{\mathbf{Z}}$ is the underlying \mathbf{Z} -local system. We assume that $L_{\mathbf{Z}}$ is torsion-free in this paper. Let \mathcal{V} be the vector bundle on S corresponding to the locally free sheaf $\mathcal{L}/F^0\mathcal{L}$, and let $\Gamma \subset \mathcal{V}$ denote the subgroup over S corresponding to the subsheaf $L_{\mathbf{Z}} \subset \mathcal{L}/F^0\mathcal{L}$. where the last injectivity follows from the condition that the weight of \mathbf{H} is negative. The family of intermediate Jacobians is defined by

$$J_S(\mathbf{H}) = \mathcal{V}/\Gamma,$$

where the quotient is set-theoretically taken fiberwise. This has a structure of a complex manifold over S .

1.2. Zucker extensions. Let \bar{S} be a partial compactification of S such that \bar{S} and $D := \bar{S} \setminus S$ are smooth. Let $j : S \rightarrow \bar{S}$ denote the inclusion. Let $\hat{\mathcal{L}}$ be the Deligne extension of \mathcal{L} such that the eigenvalues of the residues of the logarithmic connection are contained in $[0, 1)$, see [5]. By Schmid [11], the Hodge filtration F on \mathcal{L} is uniquely extended to a filtration F on $\hat{\mathcal{L}}$ such that $\mathrm{Gr}_F^p \hat{\mathcal{L}}$ are locally free (i.e. $F^p \hat{\mathcal{L}} = \hat{\mathcal{L}} \cap j_* F^p \mathcal{L}$). Let $\pi : \hat{\mathcal{V}} \rightarrow \bar{S}$ denote the vector bundle corresponding to the locally free sheaf $\hat{\mathcal{L}}/F^0 \hat{\mathcal{L}}$. Let $\hat{\Gamma}$ be the subgroup of $\hat{\mathcal{V}}$ over \bar{S} corresponding to $j_* L_{\mathbf{Z}} \subset \hat{\mathcal{L}}/F^0 \hat{\mathcal{L}}$. We define the Zucker extension by

$$J_{\bar{S}}^Z(\mathbf{H}) = \hat{\mathcal{V}}/\hat{\Gamma}.$$

We will show in Section 2 that it has a structure of a complex Lie group over \bar{S} , see [10] for the curve case.

Let $\hat{\mathcal{V}}_D$ be the restriction of $\hat{\mathcal{V}}$ over D , and $\hat{\mathcal{V}}_D^{\mathrm{Inv}}$ be the union of the subspaces defined by $\mathrm{Ker} N \subset \hat{\mathcal{L}}(s) := \hat{\mathcal{L}}/m_s \hat{\mathcal{L}}$ for $s \in D$, where m_s is the maximal ideal at s . The subspace of $J_{\bar{S}}^Z(\mathbf{H})$ defined by the quotient of $\hat{\mathcal{V}}_D^{\mathrm{Inv}}$ will be denoted by $J_{\bar{S}}^{\mathrm{GGK}}(\mathbf{H})^0$.

1.3. Admissible normal functions. A normal function is a holomorphic section ν of $J_S(\mathbf{H})$ satisfying the Griffiths transversality. By Carlson [3] it corresponds to an extension class of \mathbf{Z}_S by \mathbf{H} giving a short exact sequence

$$(1.3.1) \quad 0 \rightarrow \mathbf{H} \rightarrow \mathbf{H}' \rightarrow \mathbf{Z}_S \rightarrow 0.$$

A normal function ν is called admissible with respect to a partial compactification \bar{S} of S if \mathbf{H}' is an admissible variation of mixed Hodge structure ([9], [13]) with respect to \bar{S} , see [10]. Here \bar{S} is as in (1.2). In the unipotent monodromy case \mathbf{H}' should satisfy the following two conditions:

- (i) $\mathrm{Gr}_F^p \mathrm{Gr}_k^W \hat{\mathcal{L}}'$ are locally free.
- (ii) The relative monodromy filtration exists.

By [8] these conditions in case $w = -1$ are equivalent respectively to

- (i)' A lifting $\tilde{\nu}$ of ν in \mathcal{V} has logarithmic growth.
- (ii)' The local cohomological invariant $\gamma(\nu)$ is torsion, see also [10], Remark 1.6 (iv).

Here the local cohomological invariant $\gamma(\nu) \in H^1(S, L_{\mathbf{Z}}) = \mathrm{Ext}^1(\mathbf{Z}_S, L_{\mathbf{Z}})$ is defined by passing to the underlying short exact sequence of \mathbf{Z} -local systems of (1.3.1) after shrinking \bar{S} sufficiently, or by considering the ‘monodromy’ of a lifting $\tilde{\nu}$ of ν in \mathcal{V} , see [8].

It is known that the normal function is extended to a section of the Zucker extension if and only if the local cohomological invariant $\gamma(\nu)$ vanishes, see e.g. [10], Prop. 2.3 and 2.4 (and [7] for the geometric case). For example, if the local cohomological invariant vanishes, then the normal function is given by the difference between two splittings $\sigma_{\mathbf{Z}}, \sigma_F$ of the underlying short exact sequence of locally free sheaves of (1.3.1)

$$0 \rightarrow \hat{\mathcal{L}} \rightarrow \hat{\mathcal{L}}' \rightarrow \mathcal{O}_{\bar{S}} \rightarrow 0,$$

where $\sigma_{\mathbf{Z}}$ is defined over \mathbf{Z} and σ_F is compatible with F . Moreover we have

Theorem (Green, Griffiths, Kerr [8]). The above normal function passes through the subspace $J_{\bar{S}}^{\text{GK}}(\mathbf{H})^0$.

This is proved by using the Griffiths transversality, see [8] for details.

1.4. Limit mixed Hodge structure. From now on, assume $S = \Delta^* \times \Delta^{n-1}$ and $\bar{S} = \Delta^n$ where Δ is a polydisk. Let $t = (t_1, t_2, \dots, t_n)$ be the coordinate system of Δ^n . Set $S' = \Delta^{n-1}$ and $t' = (t_2, \dots, t_n)$.

Let $\mathbf{H} = ((\mathcal{L}, F), L_{\mathbf{Z}})$ be a polarizable variation of \mathbf{Z} -Hodge structure of weight $w < 0$, and let $(\hat{\mathcal{L}}, F)$ be as in (1.2). Assume the local monodromy T around $t_1 = 0$ is unipotent. Set $N = \log T$. By Schmid [11] we have the limit mixed Hodge structure of \mathbf{H} at 0

$$H = (H_{\mathbf{Z}}, (H_{\mathbf{Q}}, W), (H_{\mathbf{C}}; W, F)) \text{ (or } (H_{\mathbf{Z}}, W, F) \text{ to simplify the notation).}$$

By [5] (and [12] in the geometric case) $H_{\mathbf{C}}$ can be defined as

$$H_{\mathbf{C}} = \hat{\mathcal{L}}(0) (:= \hat{\mathcal{L}}/m_0\hat{\mathcal{L}}),$$

where m_0 is the maximal ideal at 0, and F is the quotient filtration of F on $\hat{\mathcal{L}}$.

For $A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}$, H_A is defined by

$$(1.4.1) \quad H_A = \Gamma(\tilde{\Delta}^* \times S', (\rho \times id)^* L_A),$$

where

$$\rho : \tilde{\Delta}^* := \{z \in \mathbf{C} \mid \text{Re } z > r_0\} \rightarrow \Delta^*$$

is a universal covering defined by $z \mapsto \exp(2\pi iz)$ (with $r_0 > 0$) and $L_A = L_{\mathbf{Z}} \otimes_{\mathbf{Z}} A$ for $A = \mathbf{Q}, \mathbf{R}$. We have the isomorphism (see [5])

$$H_{\mathbf{C}} = H_A \otimes_A \mathbf{C},$$

induced by the injection $H_A \rightarrow H_{\mathbf{C}}$ associating

$$(1.4.2) \quad \tilde{u} = \exp(-zN)u \in \hat{\mathcal{L}} \quad \text{to } u \in H_A,$$

where $z = \frac{1}{2\pi i} \log t_1$. Note that (1.4.2) gives a trivialization of the locally free sheaf $\hat{\mathcal{L}}$ (which depends on the coordinate t_1). In particular, $\{\tilde{u}_i\}$ is a basis of $\hat{\mathcal{L}}$ if $\{u_i\}$ is a basis of H_A .

The weight filtration W is given by the monodromy filtration associated to $N = \log T$ up to the shift by w . This is characterized by the two conditions: $N(W_j H_A) \subset W_{j-2} H_A$ for any j , and

$$(1.4.3) \quad N^i : \text{Gr}_{w+i}^W H_A \xrightarrow{\sim} \text{Gr}_{w-i}^W H_A \quad \text{for } i > 0.$$

If $A = \mathbf{C}$, this is compatible with F (up to a shift). We may assume for the proof of Theorem (2.1) that (1.4.3) holds also for $A = \mathbf{Z}$ replacing $L_{\mathbf{Z}}$ if necessary (using the fact that the action of N on Gr^W coincides with that of $T - id$). Note that

$$N : (H_{\mathbf{Z}}, W, F) \rightarrow (H_{\mathbf{Z}}, W, F)(-1)$$

is a morphism of mixed \mathbf{Z} -Hodge structures (see [6] for the Tate twist), and hence $K := \mathrm{Ker} N$ is a mixed \mathbf{Z} -Hodge structure of weights $\leq w$. Since $w < 0$, we have the injectivity of

$$K_{\mathbf{Z}} \rightarrow K_{\mathbf{C}}/F^0 K_{\mathbf{C}}.$$

1.5. Two splittings of the weight filtration. For $A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$, set

$$(1.5.1) \quad G_A := \bigoplus_i \mathrm{Gr}_i^W H_A \quad \text{with} \quad W_k G_A := \bigoplus_{i \leq k} \mathrm{Gr}_i^W H_A.$$

If $A = \mathbf{C}$, it has the induced filtration F . We choose and fix isomorphisms compatible with the action of N

$$(1.5.2) \quad \alpha_{\mathbf{R}} : (G_{\mathbf{R}}, W) \xrightarrow{\sim} (H_{\mathbf{R}}, W), \quad \alpha_{\mathbf{C}} : (G_{\mathbf{C}}, W, F) \xrightarrow{\sim} (H_{\mathbf{C}}, W, F),$$

inducing the identity on the graded pieces of W and such that $\alpha_{\mathbf{R}}$ is defined over \mathbf{Z} . The existence of $\alpha_{\mathbf{C}}$ is equivalent to that of a basis $(v_{i,p,k})$ of $\mathrm{Gr}_F^p P\mathrm{Gr}_{w+i}^W H_{\mathbf{C}}$ such that $v_{i,p,k}$ is represented by an element of $F^p W_{w+i} H_{\mathbf{C}}$ annihilated by N^{i+1} where $P\mathrm{Gr}_{w+i}^W$ denotes the primitive part defined by $\mathrm{Ker} N^{i+1} \subset \mathrm{Gr}_{w+i}^W$. Here the problem is the annihilation by N^{i+1} , and the existence is reduced to the surjections

$$N^{i+1} : F^p W_{w+i-1} H_{\mathbf{C}} \twoheadrightarrow F^{p-i-1} W_{w-i-3} H_{\mathbf{C}} \quad \text{for } i > 0.$$

This is further reduced to the surjections

$$N^{i+1} : F^i \mathrm{Gr}_{w+j}^W H_{\mathbf{C}} \twoheadrightarrow F^{p-i-1} \mathrm{Gr}_{w+j-2i-2}^W H_{\mathbf{C}} \quad \text{for } j \leq i-1,$$

and follows from (1.4.3). The argument is similar for $\alpha_{\mathbf{Q}}$, and it induces $\alpha_{\mathbf{R}}$. We may assume for the proof of Theorem (2.1) that $\alpha_{\mathbf{Q}}$ induces the isomorphism $\alpha_{\mathbf{Z}}$ replacing $L_{\mathbf{Z}}$ if necessary. However, $\alpha_{\mathbf{R}}$ is not compatible with $\alpha_{\mathbf{C}}$ unless W splits over A in a compatible way with the Hodge filtration F . The incompatibility is expressed by the injective morphism

$$(1.5.3) \quad \iota := \alpha_{\mathbf{C}} \circ \alpha_A^{-1} : (G_A, W) \rightarrow (G_{\mathbf{C}}, W) \quad (A = \mathbf{Z}, \mathbf{Q}, \mathbf{R}).$$

1.6. Primitive decomposition. For $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, we have the primitive decomposition

$$(1.6.1) \quad \mathrm{Gr}_i^W H_A = \bigoplus_{k \geq 0} N^k P\mathrm{Gr}_{i+2k}^W H_A,$$

where $P\text{Gr}_{w+i}^W H_A$ is the primitive part defined by $\text{Ker } N^{i+1} \subset \text{Gr}_{w+i}^W H_A$, and w is the weight of \mathbf{H} . We may assume for the proof of Theorem (2.1) that (1.6.1) holds also for $A = \mathbf{Z}$ replacing $L_{\mathbf{Z}}$ if necessary. With the notation of (1.5.1), set

$$G_{A,k}^{(j)} := N^{j-k} P\text{Gr}_{w+j}^W H_A, \quad G_A^{(j)} := \bigoplus_{0 \leq k \leq j} G_{A,k}^{(j)}, \quad G_{A,k} := \bigoplus_{k \leq j \leq m} G_{A,k}^{(j)}.$$

Then we have the bigrading

$$G_A = \bigoplus_{0 \leq j \leq m} G_A^{(j)} = \bigoplus_{0 \leq k \leq m} G_{A,k} = \bigoplus_{0 \leq k \leq j \leq m} G_{A,k}^{(j)}.$$

This is compatible with F if $A = \mathbf{C}$. Note that the $G_{A,k}$ give a splitting of the kernel filtration K_k on G_A defined by $\text{Ker } N^{k+1}$.

Since ι in (1.5.3) is compatible with the action of N , there are morphisms

$$\iota_{i,j} : G_A^{(j)} \rightarrow G_{\mathbf{C}}^{(i)},$$

such that the restriction of ι to $G_A^{(j)}$ is $\sum_i \iota_{i,j}$. Furthermore we have

$$\iota_{i,j} = \sum_k \iota_{i,j,k} \quad \text{with} \quad \iota_{i,j,k} G_{A,k'}^{(j)} \subset G_{A,k'-k}^{(i)}.$$

Here $\iota_{i,j,k}$ are compatible with the action of N , and

$$(1.6.2) \quad \iota_{i,j,k} = 0 \quad \text{if } k < 0.$$

This is equivalent to that ι preserves the kernel filtration K_k defined by $\text{Ker } N^{k+1}$.

1.7. Hodge decomposition. We have the Hodge decomposition

$$(1.7.1) \quad G_{\mathbf{C}} = \bigoplus_{i,p} (\text{Gr}_i^W H_{\mathbf{C}})^{p,i-p}.$$

This is compatible with the action of N on $G_{\mathbf{C}}$. Set

$$G_{\mathbf{C}}^{<0} := \bigoplus_{i \in \mathbf{Z}, p < 0} (\text{Gr}_i^W H_{\mathbf{C}})^{p,i-p} \subset G_{\mathbf{C}}.$$

Since the Hodge decomposition is compatible with the primitive decomposition (1.6.1), we have

$$G_{\mathbf{C}}^{<0} = \bigoplus_{0 \leq j \leq m} G_{\mathbf{C}}^{<0,(j)} = \bigoplus_{0 \leq k \leq j \leq m} G_{\mathbf{C},k}^{<0,(j)}$$

with $G_{\mathbf{C},k}^{<0,(j)} = G_{\mathbf{C}}^{<0} \cap G_{\mathbf{C},k}^{(j)}$, etc.

1.8. Trivialization of the vector bundle. Identifying $H_{\mathbf{C}}$ with $G_{\mathbf{C}}$ by $\alpha_{\mathbf{C}}$ in (1.5.2) and using the trivialization of $\widehat{\mathcal{L}}$ induced by (1.4.2), the subspace $G_{\mathbf{C}}^{<0}$ of $G_{\mathbf{C}}$ in (1.7) defines a trivialized subsheaf $\widehat{\mathcal{L}}^{<0}$ of $\widehat{\mathcal{L}}$ such that

$$(1.8.1) \quad \widehat{\mathcal{L}} = \widehat{\mathcal{L}}^{<0} \oplus F^0 \widehat{\mathcal{L}},$$

shrinking Δ if necessary. This is equivalent to the trivialization

$$(1.8.2) \quad \widehat{\mathcal{V}} = G_{\mathbf{C}}^{\leq 0} \times \bar{S}.$$

By (1.5.2), an element v of $G_{\mathbf{R}}$ is identified with an element of $H_{\mathbf{R}}$, and determines a multivalued section of $\mathcal{V} = \widehat{\mathcal{V}}|_S$. Using the trivialization (1.8.2), it is expressed as

$$(1.8.3) \quad \phi(v) \in G_{\mathbf{C}}^{\leq 0} \otimes_{\mathbf{C}} \Gamma(\tilde{S}, \mathcal{O}_{\tilde{S}}),$$

where $\tilde{S} = \tilde{\Delta}^* \times S'$ in the notation of (1.4.1). For $(z, t') \in \tilde{S}$, the value of $\phi(v)$ at (z, t') will be denoted by $\phi(v; z, t')$.

1.9. Norms. We choose and fix a norm on each complex vector space $G_{\mathbf{C},k}^{\leq 0,(j)}$, i.e. there is a continuous map $v \mapsto |v| \in \mathbf{R}_{\geq 0}$ satisfying the usual relations: $|v + v'| \leq |v| + |v'|$, $|av| = |a||v|$ for $a \in \mathbf{C}$, and $\{|v| = 1\}$ is compact. We may assume that it is compatible with the Hodge decomposition (1.7.1) (i.e. the norm is the sum of the norm of the direct factors). This induces a norm on $G_{\mathbf{C}}^{\leq 0}$ compatible with the primitive and Hodge decompositions.

For $v = \sum_{0 \leq k \leq j \leq m} v_k^{(j)} \in G_{\mathbf{C}}^{\leq 0}$ with $v_k^{(j)} \in G_{\mathbf{C},k}^{\leq 0,(j)}$, set

$$(1.9.1) \quad |v|_k := |v_k| = \sum_{0 \leq j \leq m} |v_k^{(j)}| \quad \text{with} \quad v_k = \sum_{k \leq j \leq m} v_k^{(j)}.$$

We also choose and fix a norm on each real vector space $P\text{Gr}_{w+i}^W H_{\mathbf{R}}$. This induces a norm on G_A using the primitive decomposition (1.6.1). Let

$$u = \sum_{0 \leq k \leq i \leq m} u_k^{(i)} = \sum_{0 \leq k \leq m} u_k \in G_{\mathbf{R}},$$

where $u_k^{(i)} = N^{i-k} u_{i,k}$ with $u_{i,k} \in P\text{Gr}_{w+i}^W H_{\mathbf{R}}$ and $u_k = \sum_{k \leq i \leq m} u_k^{(i)}$. Let $z \in \tilde{\Delta}^*$, and set

$$(1.9.2) \quad \begin{aligned} E(G_{\mathbf{Z}}) &:= \min\{|u| : u \in G_{\mathbf{Z}} \setminus \{0\}\}, \quad y := \text{Im } z, \\ A(u, z) &:= \sum_{0 \leq k \leq m} |u_k| y^k = \sum_{0 \leq k \leq i \leq m} |u_{i,k}| y^k, \\ B(u, z) &:= \sum_{1 \leq k \leq m} |u_k| y^{k-1} = \sum_{1 \leq k \leq i \leq m} |u_{i,k}| y^{k-1}. \end{aligned}$$

For $\eta > 0$ and $r > 1$, set

$$(1.9.3) \quad I_{r,\eta} = \{(z, t') \in \mathbf{C} \times S' \mid 0 \leq \text{Re } z < 1, \text{Im } z > r, |t_j| < \eta (j > 1)\} \subset \tilde{S}.$$

In the case $S' = pt$, we will denote $I_{r,\eta}$ by I_r .

1.10. Examples. (i) Assume $H_{\mathbf{Z}} = \mathbf{Z}e_0 + \mathbf{Z}e_1$ with $Ne_0 = e_1$, $Ne_1 = 0$, and $F^0 = \mathcal{O}_{\Delta} \tilde{e}_0$ (i.e. an \mathbf{R} -split nilpotent orbit). Then $\widehat{\mathcal{V}} = \mathbf{C} \times \Delta$, and e_0, e_1 are identified respectively with the (multivalued) functions 1 and $z := \frac{1}{2\pi i} \log t$ on Δ^* . (This is the simplest example with multivalued

Examples 2. $H_{\mathbf{Z}} = \mathbf{Z}e_0 + \mathbf{Z}e_1 + \mathbf{Z}f_0 + \mathbf{Z}f_1$ s.t.

$$Ne_0 = e_1, Ne_1 = 0, Nf_0 = f_1, Nf_1 = 0,$$

$$F^0 = \mathcal{O}_\Delta \tilde{v}_0 + \mathcal{O}_\Delta \tilde{v}_1 \text{ with } v_k = e_k - if_k \text{ } (k = 0, 1).$$

Then $\widehat{\mathcal{V}} = \mathbf{C}^2 \times \Delta$ s.t. $\Gamma \Leftrightarrow \{(az + b, a) \mid a, b \in \mathbf{Z}[i]\}$.

So $\widehat{\mathcal{V}}/\widehat{\Gamma}$ is not Hausdorff.

Indeed, $(\gamma, a) \in \overline{\Gamma} \subset \widehat{\mathcal{V}}$ for any $\gamma \in \mathbf{C}$, $a \in \mathbf{Z} \setminus \{0\}$, considering $(az - ni, a)$ for $z = (\gamma + ni)/a$ ($n \gg 1$).

2. Norm estimates

In this Section we prove Theorems 1 and 2. We first show the following which is a key to the proof of Theorem 2.

2.1. Theorem. *With the notation of (1.5–9), let $v \in G_{\mathbf{C}}^{<0} \cap \text{Ker } N$. Then there exist $\varepsilon, \eta > 0$ and $r > 1$ such that for any $u = \sum_{j,k} u_k^{(j)} \in G_{\mathbf{Z}} \setminus \text{Ker } N$ and for $(z, t') \in I_{r,\eta}$, the following holds:*

(i) *We have for some $k \geq 0$*

$$(2.1.1) \quad |\phi(\iota(u); z, t') - v|_k \geq \varepsilon A(u, z)/y^k.$$

(ii) *If (2.1.1) holds for some $k > 0$, then $u_{k'}^{(j)} \neq 0$ for some $k' \geq k$.*

2.2. Remarks. (i) Condition (2.1.1) is essentially independent of the choice of norms on $G_{\mathbf{C},k}^{<0,(j)}$ and $P\text{Gr}_{w+i}^W H_{\mathbf{R}}$, replacing ε if necessary.

(ii) If $u \in \text{Ker } N$ and (2.1.1) does not hold for every k , then

$$\iota(u) - v \in F^0 G_{\mathbf{C}},$$

assuming that ε is sufficiently small compared to the distance between the images of v and $\iota(G_{\mathbf{Z}} \cap \text{Ker } N) \setminus \{v\}$ in $G_{\mathbf{C}}/F^0 G_{\mathbf{C}}$.

2.3 Proof of Theorem (2.1)(i). We start with the following

(a) Reduction to the case of a nilpotent orbit with $S' = pt$.

Using the trivialization of the vector bundle $\widehat{\mathcal{L}}$ given by (1.4.2), there is

$$M(t) \in \text{Hom}_{\mathbf{C}}(F^0 G_{\mathbf{C}}, G_{\mathbf{C}}^{<0}) \otimes_{\mathbf{C}} \Gamma(\bar{S}, \mathcal{O}_{\bar{S}}),$$

such that the fiber of $F^0 \widehat{\mathcal{L}}$ at $t \in \bar{S}$ is expressed by the graph of $M(t)$ using the following direct sum decomposition in the notation of (1.7):

$$G_{\mathbf{C}} = G_{\mathbf{C}}^{<0} \oplus F^0 G_{\mathbf{C}}.$$

Note that $M(0) = 0$. Here $G_{\mathbf{C}}$ is identified with $H_{\mathbf{C}}$ by $\alpha_{\mathbf{C}}$ in (1.5.2).

By the inverse of (1.4.2), $\tilde{v} \in G_{\mathbf{C}}$ corresponds to a multivalued section v of \mathcal{L} , which is expressed as

$$(2.3.1) \quad \sum_{k \geq 0} N^k \tilde{v} \otimes z^k / k! \in G_{\mathbf{C}} \otimes_{\mathbf{C}} \Gamma(\tilde{S}, \mathcal{O}_{\tilde{S}}).$$

Using the above decomposition, its image in $\widehat{\mathcal{V}}$ (see (1.8.3)) is given by

$$\begin{aligned} \phi(v) &= \phi'(v) - M(t)\phi''(v) \in G_{\mathbf{C}}^{\leq 0} \otimes_{\mathbf{C}} \Gamma(\tilde{S}, \mathcal{O}_{\tilde{S}}) \quad \text{with} \\ \phi'(v) &= \sum_{k \geq 0} pr_1(N^k \tilde{v}) \otimes z^k / k!, \quad \phi''(v) = \sum_{k \geq 0} pr_2(N^k \tilde{v}) \otimes z^k / k!, \end{aligned}$$

where $pr_1 : G_{\mathbf{C}} \rightarrow G_{\mathbf{C}}^{\leq 0}$ and $pr_2 : G_{\mathbf{C}} \rightarrow F^0 G_{\mathbf{C}}$ are the projections associated to the above decomposition. By (1.6.2) we have

$$|\phi''(\iota(u); z, t')|_k \leq CA(u, z)/y^k \leq CA(u, z) \quad \text{for any } 0 \leq k \leq m,$$

where $C > 0$ is independent of u, z, t' (and $1 \leq y \leq |z| \leq 2y$ because $r \geq 1$). Since $M(0) = 0$, this implies

$$(2.3.2) \quad |M(t)\phi''(\iota(u); z, t')|_k \leq C'(\sum_{j \geq 0} |t_j|)A(u, z) \quad \text{for any } 0 \leq k \leq m,$$

where $C' > 0$ depends only on $C, M(t)$, and is independent of u, z, t' . Then

$$|M(t)\phi''(\iota(u); z, t')|_k \leq \varepsilon A(u, z)/y^k \quad \text{for any } 0 \leq k \leq m,$$

if $1/r$ and η are sufficiently small (depending only on C', m and ε). So we may replace ϕ and ε in (2.1.1) by ϕ' and 2ε respectively. Thus the assertion is reduced to the case \mathbf{H} is a nilpotent orbit, where $M(t) = 0$ and $\phi'' = 0$. (Note that a variation of Hodge structure is a nilpotent orbit if and only if there is a basis $\{v_i\}$ of $H_{\mathbf{Z}} \otimes \mathbf{C}$ such that the \tilde{v}_i in (1.4.2) give generators of $\bigoplus_p \text{Gr}_F^p \widehat{\mathcal{L}}$.) Then we may assume moreover

$$S' = pt, \quad S = \Delta^*.$$

From now on, $\phi(\iota(u); z, t')$ will be denoted by $\phi(\iota(u), z)$.

(b) Reduction to the case: $\iota_{i,j,k} = 0$ for $k \neq 0$.

For any $u^{(j)} \in G_{\mathbf{Z}}^{(j)}$ and $z \in I_r$, we have

$$(2.3.3) \quad |\phi(\iota_{i,j,k}(u^{(j)}), z)|_{k'} \leq C'' A(u^{(j)}, z)/y^{k+k'},$$

where $C'' > 0$ is independent of $u^{(j)}, z$. So the assertion is reduced to the desired case by the same argument as above. In particular, we get

$$\iota_{i,j} = 0 \quad \text{if } i < j,$$

and $\iota_{i,j}$ are graded morphisms (up to the shift by $i - j$).

(c) **Reduction to the case:** $G_{\mathbf{R}} = G_{\mathbf{R}}^{(i)}$.

Let $v = \sum_i v^{(i)}$ be the decomposition such that $v^{(i)} \in G_{\mathbf{C}}^{(i)} \cap \text{Ker } N$. Assume the assertion is true for each $(G_{\mathbf{R}}^{(i)}, G_{\mathbf{C}}^{(i)}, \iota_{i,i}, v^{(i)})$. Then there exist $\varepsilon', r' > 0$ such that for any $u^{(i)} \in G_{\mathbf{Z}}^{(i)} \setminus \text{Ker } N$ and $z \in I_{r'}$, we have

$$(2.3.4) \quad |\phi(\iota_{i,i}(u^{(i)}), z) - v^{(i)}|_{k(i)} \geq \varepsilon' A(u^{(i)}, z)/y^{k(i)} \quad \text{for some } k(i) \geq 0,$$

where $k(i)$ may depend on i . However, ε', r' are independent of i (taking the minimum of ε and the maximum of r for i).

Assume on the other hand the assertion is false for $(G_{\mathbf{R}}, G_{\mathbf{C}}, \iota, v)$. Then, for any $\varepsilon, r > 0$, there exist $u = \sum_i u^{(i)} \in G_{\mathbf{Z}} \setminus \text{Ker } N$ and $z \in I_r$ such that

$$|\phi(\iota(u), z) - v|_k \leq \varepsilon A(u, z)/y^k \quad \text{for any } k \geq 0.$$

By (2.3.3) with $k = 0$ and k' replaced by k , we get

$$|\phi(\iota_{i,j}(u^{(j)}), z)|_k \leq C_{i,j} A(u^{(j)}, z)/y^k \quad \text{for any } i, j, k,$$

where $C_{i,j} > 0$ is independent of $u^{(j)}, z$. We have furthermore for any i, j, k

$$|\phi(\iota_{i,i}(u^{(i)}), z) - v^{(i)}|_k \leq |\phi(\iota(u), z) - v|_k + \sum_{j < i} |\phi(\iota_{i,j}(u^{(j)}), z)|_k.$$

In case (2.3.4) holds for i , we get thus

$$(2.3.5) \quad \varepsilon' A(u^{(i)}, z) < \varepsilon A(u, z) + \sum_{j < i} C_{i,j} A(u^{(j)}, z).$$

Here we do not necessarily assume $u^{(i)} \notin \text{Ker } N$. Note that (2.3.4) does not hold only in the case $u^{(i)} \in \text{Ker } N$ with $\iota_{i,i}(u^{(i)}) = v^{(i)} \bmod F^0$ (assuming ε' sufficiently small). In this case we have $A(u^{(i)}, z) = |u^{(i)}|$, and this can be bounded essentially by $|v^{(i)}|$ using the injectivity of

$$G_{\mathbf{Z}} \cap \text{Ker } N \rightarrow G_{\mathbf{C}}/F^0.$$

Since $u \in G_{\mathbf{Z}} \setminus \text{Ker } N$, we see that (2.3.5) holds also in this case if we assume in the notation of (1.9.2) r is sufficiently large compared to

$$\varepsilon' |v^{(i)}| / \varepsilon E(G_{\mathbf{Z}}).$$

Thus (2.3.5) holds for any i . This contradicts the following if ε is sufficiently small, where $a_i = A(u^{(i)}, z)$, $C'_{i,j} = C_{i,j}/\varepsilon'$, and $\varepsilon'' = \varepsilon/\varepsilon'$.

Sublemma. *Let $C'_{i,j}$ be positive numbers for $0 \leq j < i \leq m$. Then there exists a positive number $\varepsilon'' < 1/2$ such that for any nonnegative numbers a_i ($0 \leq i \leq m$), the following condition implies $a_i = 0$:*

$$a_i \leq \varepsilon'' (\sum_j a_j) + \sum_{j < i} C'_{i,j} a_j \quad \text{for } 0 \leq i \leq m.$$

Proof. The condition for $i = 0$ implies

$$a_0 \leq 2\varepsilon''(\sum_{j>0} a_j),$$

and hence

$$a_i \leq B\varepsilon''(\sum_{j>0} a_j) + \sum_{0<j<i} C'_{i,j} a_j \quad \text{for } 1 \leq i \leq m,$$

where $B > 0$ depends only on the $C'_{i,0}$. So the assertion follows by induction on m .

Thus the assertion is reduced to the case $G_{\mathbf{R}} = G_{\mathbf{R}}^{(i)}$.

(d) Reduction to the case: $\dim P\mathrm{Gr}_{w+i}^W H_{\mathbf{R}} = 1$ or 2 .

We replace the assertion by the following which implies Theorem (2.1)(i) by the above arguments, since $u \in G_{\mathbf{Z}}$ belongs to $\mathrm{Ker} N$ if we have in the notation of (1.9.2)

$$B(u, z) \leq A(u, z)/r \leq C_v/r < E(G_{\mathbf{Z}}).$$

2.4. Proposition. *With the above notation, assume \mathbf{H} is a nilpotent orbit on $S = \Delta^*$, $G_{\mathbf{R}} = G_{\mathbf{R}}^{(i)}$ for some i , and $\iota_{i,j,k} = 0$ for $k \neq 0$. Then, for $v \in G_{\mathbf{C}}^{\leq 0} \cap \mathrm{Ker} N$, there exist $C_v, \varepsilon > 0$ and $r > \max(C_v/E(G_{\mathbf{Z}}), 1)$ such that for any $u = \sum_{j,k} u_k^{(j)} \in G_{\mathbf{R}}$ and for $z \in I_r$, we have*

$$A(u, z) \leq C_v,$$

if the following condition is satisfied for any $k \geq 0$:

$$(2.4.1) \quad |\phi(\iota(u), z) - v|_k \leq \varepsilon A(u, z)/y^k.$$

Proof. Since the assertion depends only on the underlying variation of \mathbf{R} -Hodge structure, we may assume that \mathbf{H} is defined only over \mathbf{R} (forgetting the \mathbf{Q} -structure). Furthermore, if $\mathbf{H} = \mathbf{H}' \oplus \mathbf{H}''$ and the assertion holds for $\mathbf{H}', \mathbf{H}''$, then the assertion also holds for \mathbf{H} (modifying ε if necessary). Indeed, for $u = u' + u''$ and $v = v' + v''$ with u', v' coming from \mathbf{H}' and similarly for u'', v'' , we have

$$|\phi(\iota(u'), z) - v'|_k \leq |\phi(\iota(u), z) - v|_k \leq 2\varepsilon A(u', z)/y^k,$$

if $A(u', z) \geq A(u'', z)$. Thus the assertion is reduced to the case $\dim P\mathrm{Gr}_{w+i}^W H_{\mathbf{R}} = 1$ or 2 . In the two-dimensional case, take a basis (v, \bar{v}) of $P\mathrm{Gr}_{w+i}^W H_{\mathbf{C}}$ which is compatible with the Hodge decomposition. Then $av + a'\bar{v}$ is real if and only if $\bar{a} = a'$. Using (2.3.1), the image of $u = \sum_{0 \leq k \leq i} N^{i-k} u_k \in G_{\mathbf{R}}$ is then expressed as

$$(d^k f(z)/dz^k, d^k \bar{f}(z)/dz^k)_{0 \leq k \leq i},$$

if $f(z) = \sum_{0 \leq k \leq i} a_k z^k/k! \quad \text{with} \quad a_k v + \bar{a}_k \bar{v} = u_k,$

where $\bar{f}(z) = \overline{f(\bar{z})}$. So the assertion is reduced to the following (where we assume $f = \bar{f}$ and $n_1 = n_2$ in the case $\dim P\mathrm{Gr}_{w+j}^W H_{\mathbf{R}} = 1$).

2.5 Lemma. Let n, n_1, n_2 be nonnegative integers such that $n_1 + n_2 > n$. Let $\mathbf{C}[z]^{\leq n}$ be the set of polynomials of degree $\leq n$. For $f \in \mathbf{C}[z]$ and $z_0 \in \mathbf{C}$, set $y_0 := \text{Im } z_0$, and

$$A(f, z_0) := \sum_{k \geq 0} |(d^k f / dz^k)(0)| y_0^k.$$

Then for any $a, a' \in \mathbf{C}$, there exist $C_{a, a'}, \varepsilon > 0, r > 1$ such that for any $f(z) \in \mathbf{C}[z]^{\leq n}$ and $z_0 \in I_r$, we have

$$A(f, z_0) \leq C_{a, a'},$$

if the following conditions are satisfied.

$$(2.5.1) \quad \begin{aligned} |(d^k(f - a)/dz^k)(z_0)| &\leq \varepsilon A(f, z_0)/y_0^k \quad \text{for any } 0 \leq k < n_1, \\ |(d^k(\bar{f} - a')/dz^k)(z_0)| &\leq \varepsilon A(f, z_0)/y_0^k \quad \text{for any } 0 \leq k < n_2. \end{aligned}$$

Proof. Note first that $y_0 \leq |z_0| \leq 2y_0$ since $r > 1$. We may assume $n_1 \geq n_2$ exchanging f and \bar{f} if necessary. For $f \in \mathbf{C}[z]$, set

$$b_k = (d^k(f - a)/dz^k)(z_0).$$

There is $g(z) \in \mathbf{C}[z]^{\leq n - n_1}$ such that

$$f(z) - a = \sum_{0 \leq k < n_1} (b_k/k!)(z - z_0)^k + g(z)(z - z_0)^{n_1}.$$

Using the binomial coefficients, we see that

$$(2.5.2) \quad A((z - z_0)^k, z_0) = A((z - \bar{z}_0)^k, z_0) \leq C(k)y_0^k,$$

where $C(k) > 0$ depends only on k . Applying the Leibniz rule to $g(z)(z - z_0)^{n_1}$ and using (2.5.1), we then get

$$A(f - a, z_0) \leq \varepsilon C_1 A(f, z_0) + C_2 A(g, z_0) y_0^{n_1},$$

where $C_1, C_2 > 0$ (depending only on n, n_1). This implies

$$(2.5.3) \quad (1 - \varepsilon C_1) A(f, z_0) \leq C_2 A(g, z_0) y_0^{n_1} + |a|.$$

On the other hand, the above equation can be modified as

$$\bar{g}(z) = (z - \bar{z}_0)^{-n_1} ((\bar{f}(z) - a') + (a' - \bar{a})) - \sum_{0 \leq k < n_1} (\bar{b}_k/k!)(z - \bar{z}_0)^{k - n_1}.$$

Then (2.5.1) implies similarly

$$|(d^k \bar{g}/dz^k)(z_0)| \leq \varepsilon C_3 A(f, z_0)/y_0^{k+n_1} + C_4(|a| + |a'|)/y_0^{k+n_1} \quad \text{for } 0 \leq k < n_2,$$

where $C_3, C_4 > 0$ (depending only on n, n_1). Assuming $\varepsilon \leq (2C_1)^{-1}$ and using (2.5.3), we then get

$$|(d^k \bar{g}/dz^k)(z_0)| \leq \varepsilon C_5 A(g, z_0)/y_0^k + C_6(|a| + |a'|)/y_0^{k+n_1} \quad \text{for } 0 \leq k \leq n - n_1,$$

where $C_5, C_6 > 0$ (depending only on n, n_1). Combined with (2.5.2), this implies

$$A(g, z_0) \leq \varepsilon C_7 A(g, z_0) + C_8(|a| + |a'|)/y_0^{n_1} \quad \text{for } 0 \leq k \leq n - n_1,$$

where $C_7, C_8 > 0$ (depending only on n, n_1). Assuming $\varepsilon \leq (2C_7)^{-1}$, we deduce from this and (2.5.3)

$$A(f, z_0) \leq C_9(|a| + |a'|),$$

where $C_9 > 0$ (depending only on n, n_1). So the assertion follows. This completes the proofs of Lemma (2.5), Proposition (2.4) and Theorem (2.1)(i).

2.6. Proof of Theorem (2.1)(ii). We have $v_k = 0$ since $k > 0$ and $v \in \text{Ker } N$. So it is enough to show

$$|\phi(\iota(u); z, t')|_k \leq \varepsilon A(u, z)/y^k \quad \text{if } u_{k'}^{(j)} = 0 \text{ for } k' \geq k.$$

Then the assertion is reduced to the nilpotent orbit case by (2.3.2) (replacing $\varepsilon, \varepsilon', r$ if necessary). In this case we have

$$|\phi(\iota(u); z, t')|_k = 0 \quad \text{if } u_{k'}^{(j)} = 0 \text{ for } k' \geq k,$$

because $\iota_{i,j,k} = 0$ for $k < 0$ by (1.6.2). So the assertion follows.

From Theorem (2.1) we can deduce

2.7. Corollary. *For any $p \in \widehat{\mathcal{V}}_D^{\text{Inv}}$, there is a sufficiently small open neighborhood U_p of p in $\widehat{\mathcal{V}}$ such that*

$$(2.7.1) \quad U_p \cap \widehat{\Gamma} \subset \widehat{\Gamma}(p),$$

where $\widehat{\Gamma}(p)$ is the section of $\widehat{\Gamma}$ passing through p if $p \in \widehat{\Gamma}$ and $\widehat{\Gamma}(p) = \emptyset$ otherwise.

Proof. In case $p \in \widehat{\Gamma}$ we may assume $p = 0$ using the action of $\widehat{\Gamma}(p)$ on $\widehat{\mathcal{V}}$. Take $v \in G_{\mathbf{C}}^{<0} \cap \text{Ker } N$ corresponding to p , and apply Theorem (2.1). Set

$$U_p = \{(v', t) \in G_{\mathbf{C}}^{<0} \times \Delta^m \mid |v' - v| < \varepsilon, |t_1| < e^{2\pi r}, |t_j| < \eta (j > 1)\},$$

where ε, η, r are as in Theorem (2.1). Then, for any $u \in G_{\mathbf{Z}} \setminus \text{Ker } K$, (2.1.1) is satisfied for some $k \geq 0$. So the assertion follows (since $r > 1$).

2.8. Proof of Theorem 2. For the first assertion we have to show that for any $p \in \widehat{\mathcal{V}}_D$ in the notation of (1.2), there is an open neighborhood U_p in $\widehat{\mathcal{V}}$ such that

$$(U_p - U_p) \cap \widehat{\Gamma} \subset \widehat{\Gamma}(0),$$

where $\widehat{\Gamma}(0)$ denotes the zero section. So it is enough to show that for $0 \in \widehat{\Gamma}(0)$ over $0 \in \bar{S}$, there is an open neighborhood U_0 of 0 in $\widehat{\mathcal{V}}$ such that

$$(2.8.1) \quad U_0 \cap (\widehat{\Gamma} \setminus \widehat{\Gamma}(0)) = \emptyset.$$

Then the assertion follows from Corollary (2.7).

For the second assertion we have to show that for any $p_1, p_2 \in \widehat{\mathcal{V}}_D^{\text{Inv}}$ such that $\pi(p_1) = \pi(p_2)$ and $p := p_1 - p_2 \notin \widehat{\Gamma}$, there are open neighborhoods U_i of p_i in $\widehat{\mathcal{V}}$ such that

$$(U_1 - U_2) \cap \widehat{\Gamma} = \emptyset \quad \text{if } p \notin \widehat{\Gamma},$$

where $\widehat{\Gamma}(p) \subset \widehat{\Gamma}$ is the section over \bar{S} passing p (shrinking \bar{S} if necessary). So it is enough to show that for any $p \in \widehat{\mathcal{V}}_D^{\text{Inv}} \setminus \widehat{\Gamma}$, there is an open neighborhood U_p of p in $\widehat{\mathcal{V}}$ such that

$$(2.8.2) \quad U_p \cap \widehat{\Gamma} = \emptyset.$$

Then the assertion follows from Corollary (2.7).

2.9. Proof of Theorem 1. Taking a finite covering, the assertion is reduced to the case where the local monodromy is unipotent. Let ν be an admissible normal function such that the local cohomological invariant $\gamma(\nu)$ is torsion, see (1.3). Then there is a normal function μ whose associated exact sequence (1.3.1) splits after tensoring with \mathbf{Q} and such that

$$\gamma(\nu) = \gamma(\mu).$$

Let $\tilde{\nu}, \tilde{\mu}$ be multivalued holomorphic sections of \mathcal{V} lifting ν and μ respectively. Since μ vanishes by the scalar extension $\mathbf{Z} \rightarrow \mathbf{Q}$, there is a positive integer m such that $m\mu = 0$, i.e. $\tilde{\mu}$ is a multivalued section of $\frac{1}{m}\Gamma$. Set

$$\nu' = \nu - \mu, \quad \tilde{\nu}' = \tilde{\nu} - \tilde{\mu}.$$

For the study of the zero locus of ν , it is then sufficient to consider the intersection

$$(2.9.1) \quad \tilde{\nu}' \cap (\Gamma - \tilde{\mu}) \quad (\subset \tilde{\nu}' \cap \frac{1}{m}\Gamma),$$

instead of $\tilde{\nu} \cap \Gamma$. Here a section is identified with its image. Since $\gamma(\nu') = 0$, we may assume that $\tilde{\nu}'$ is univalent and is extended to a section of $\widehat{\Gamma}$. As explained in (1.3), we have by a recent theorem of Green, Griffiths and Kerr [8]

$$\tilde{\nu}'(0) \in J_{\bar{S}}^{\text{GK}}(\mathbf{H})^0.$$

Then, applying Corollary (2.7) to $\tilde{\nu}'(0)$ and $\frac{1}{m}\widehat{\Gamma} \subset \widehat{\mathcal{V}}$, there is a sufficiently small open neighborhood $U_{\tilde{\nu}'(0)}$ of $\tilde{\nu}'(0)$ such that

$$(2.9.2) \quad \frac{1}{m}\widehat{\Gamma} \cap U_{\tilde{\nu}'(0)} = \begin{cases} \tilde{\mu}' \cap U_{\tilde{\nu}'(0)} & \text{if } \tilde{\nu}'(0) \in \frac{1}{m}\widehat{\Gamma}, \\ \emptyset & \text{if } \tilde{\nu}'(0) \notin \frac{1}{m}\widehat{\Gamma}, \end{cases}$$

where $\tilde{\mu}'$ is the section of $\frac{1}{m}\hat{\Gamma}$ passing through $\tilde{\nu}'(0) \in \frac{1}{m}\hat{\Gamma}$.

If $\tilde{\nu}'(0) \notin \frac{1}{m}\hat{\Gamma}$ or $\tilde{\nu}'(0) \in \frac{1}{m}\hat{\Gamma}$ with $\gamma(\nu) \neq 0$, then (2.9.1) is empty (shrinking \bar{S} if necessary) since $\tilde{\mu}'$ is not a section of $\Gamma - \tilde{\mu}$ in the last case. (Indeed, if $\tilde{\mu}' + \tilde{\mu}$ is a multivalued section of Γ , then $\tilde{\mu}'$ coincides with $-\tilde{\mu}$ replacing $\tilde{\mu}$ if necessary, and hence $\tilde{\mu}$ is univalent. But this contradicts the condition that $\gamma(\mu) = \gamma(\nu) \neq 0$.) This implies that the zero locus of ν is empty on a neighborhood of 0.

If $\tilde{\nu}'(0) \in \frac{1}{m}\hat{\Gamma}$ with $\gamma(\nu) = 0$, then we may assume $\tilde{\mu} = 0$, $\tilde{\nu}' = \tilde{\nu}$, $m = 1$, and furthermore $\tilde{\nu}(0) = 0$ replacing $\tilde{\nu}$ if necessary. So the assertion follows from (2.9.2).

This completes the proof of Theorem 1.

2.11. Proof of Theorem 3. By definition [8] (see also [10]), the Néron model $J_S^{\text{GK}}(\mathbf{H})$ is the union of $Y_\nu := J_S^{\text{GK}}(\mathbf{H})^0|_{U(\nu)}$, where ν runs over normal functions defined over open subsets $U(\nu) \setminus D$ which are admissible with respect to $U(\nu)$. We have the identifications

$$Y_\nu|_{U(\nu) \cap U(\nu') \setminus D} \cong Y_{\nu'}|_{U(\nu) \cap U(\nu') \setminus D},$$

induced by $\nu - \nu'$ over $U(\nu) \cap U(\nu') \setminus D$ if the cohomology classes of ν and ν' are different. In case they coincide, then the isomorphism is extended over $U(\nu) \cap U(\nu')$. Moreover we may assume that the ν are torsion since $w = -1$. So it is enough to show that for any $p_1, p_2 \in \hat{\mathcal{V}}_D^{\text{Inv}}$ with $\pi(p_1) = \pi(p_2)$, there exists respectively open neighborhoods U_1, U_2 of p_1, p_2 in $\hat{\mathcal{V}}$ such that

$$(U_1 - U_2) \cap \frac{1}{m}\hat{\Gamma}$$

is contained in the sections of $\frac{1}{m}\hat{\Gamma}$ over \bar{S} shrinking \bar{S} if necessary. Then the assertion follows from Corollary (2.7).

2.12. Remark. The norm in (2.1.1) uses the trivialization (1.8.2), and is quite different from the one in [11] and [4] using the Weil operator at each point of S . It does not seem easy to generalize the method in this paper to the case D is a divisor with normal crossings since at least the multi SL_2 -orbit theorem [4] will be needed. Note that this case is treated in [2].

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